The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Homework 6
Please submit your assignment online on Blackboard
Due at 18:00, Mar.17, 2025

1. Let $S = \{1, 2\}$ be the state space of a Markov chain $X = (X_n)_{n\geq 0}$, with transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}.$$

Let the initial distribution of the Markov chain be given by $\mu = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, i.e. $\mathbb{P}[X_0 = 1] = \mathbb{P}[X_0 = 2] = 0.5$.

(a) Find eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and eigenvectors $v_1, v_2 \in \mathbb{R}^2$ of the matrix P,

$$\text{i.e. } Pv_1 = \lambda_1 v_1 \quad \text{and} \quad Pv_2 = \lambda_2 v_2.$$

(b) Find the matrices V and V^{-1}, such that $VV^{-1} = I_2$, and

$$P = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1},$$

where I_2 denotes the 2×2 identity matrix, and V^{-1} is the inverse of matrix V.

(c) Let μ_n denote the law of X_n, i.e. $\mathbb{P}[X_n = 1] = \mu_n(1)$ and $\mathbb{P}[X_n = 2] = \mu_n(2)$. Compute

$$P^n \quad \text{and then} \quad \mu_n := \mu^\top P^n.$$

(d) Compute

$$P^\infty := \lim_{n\to\infty} P^n \text{ and } \mu_\infty^\top := \mu^\top P^\infty = \lim_{n\to\infty} \mu^\top P^n.$$

Then deduce that

$$\lim_{n\to\infty} \mathbb{P}[X_n = 1] = \frac{3}{8} \text{ and } \lim_{n\to\infty} \mathbb{P}[X_n = 2] = \frac{5}{8}.$$

(e) Verify that $\mu_\infty^\top P = \mu_\infty^\top$.

(f) Assume that $\mathbb{P}[X_0 = 1] = \frac{3}{8}$ and $\mathbb{P}[X_0 = 2] = \frac{5}{8}$, prove that

$$\mathbb{P}[X_1 = 1] = \frac{3}{8} \text{ and } \mathbb{P}[X_2 = 2] = \frac{5}{8}.$$

(**Remark:** the measure (vector) $\mu_\infty = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}$ is called the invariant measure of the Markov chain.)

Solution

1. (a) The characteristic equation is

$$\det(P - \lambda I_2) = \begin{vmatrix} 0.5-\lambda & 0.5 \\ 0.3 & 0.7-\lambda \end{vmatrix}$$
$$= (2\lambda - 2)(0.5\lambda - 0.1) = 0$$

Then

$$\lambda_1 = 1, \qquad \lambda_2 = 0.2.$$

Thus, the eigenvalues are 1 and 0.2.
Let

$$v = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Substitute $\lambda = 1$ in the equation:

$$(P - \lambda I_2)v = 0$$

yields

$$\begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving above equation system, we have $x = y$. Hence the corresponding eigenvector is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 0.2$, we have

$$\begin{bmatrix} 0.3 & 0.5 \\ 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies that $y = -0.6x$. It follows that the corresponding eigenvector is

$$v_2 = \begin{bmatrix} 1 \\ -0.6 \end{bmatrix}.$$

(b) Let

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.6 \end{bmatrix}.$$

Then

$$V^{-1} = -\frac{1}{1.6} = \begin{bmatrix} -0.6 & -1 \\ -1 & 1 \end{bmatrix}.$$

Note that

$$PV = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 1 & -0.12 \end{bmatrix}$$

and

$$V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 0 & -0.12 \end{bmatrix}$$

Hence

$$PV = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and

$$P = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1}.$$

(c) Let

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Applying result from (b), we have

$$\begin{aligned} P^n &= VDV^{-1}VDV^{-1} \cdots VDV^{-1} = VD^nV^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -0.6 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (0.2)^n \end{bmatrix} (-\frac{1}{1.6}) \begin{bmatrix} -0.6 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{1.6} \begin{bmatrix} 0.6 + (0.2)^n & 1 - (0.2)^n \\ 0.6 - 0.6(0.2)^n & 1 + 0.6(0.2)^n \end{bmatrix}. \end{aligned}$$

$$\mu^\top P^n = \frac{1}{1.6} \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 + (0.2)^n & 1 - (0.2)^n \\ 0.6 - 0.6(0.2)^n & 1 + 0.6(0.2)^n \end{bmatrix} = \frac{1}{1.6} \begin{bmatrix} 0.6 + 0.2(0.2)^n & 1 - 0.2(0.2)^n \end{bmatrix}$$

(d) Using the fact that

$$\lim_{n \longrightarrow \infty} (0.2)^n = 0$$

to derive

$$P^\infty := \lim_{n \longrightarrow \infty} P^n = \frac{1}{1.6} \begin{bmatrix} 0.6 & 1 \\ 0.6 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

$$\mu_\infty^\top := \mu^\top P^\infty = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{3}{8} & \frac{5}{8} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}.$$

Hence

$$\lim_{n \to \infty} \mathbb{P}[X_n = 1] = \mu_\infty^\top(1) = \frac{3}{8}$$

and

$$\lim_{n \to \infty} \mathbb{P}[X_n = 2] = \mu_\infty^\top(2) = \frac{5}{8}.$$

(e)

$$\mu_\infty^\top P = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix} = \mu_\infty^\top.$$

(f) Given

$$\mathbb{P}[X_0 = 1] = \frac{3}{8} \text{ and } \mathbb{P}[X_0 = 2] = \frac{5}{8}.$$

Then

$$X_0 \sim \mu_\infty^\top$$

Applying (e), we have

$$\begin{aligned} X_1 &\sim \mu_\infty^\top P = \mu_\infty^\top \\ X_2 &\sim \mu_\infty^\top P = \mu_\infty^\top. \end{aligned}$$